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FINITE DEFORMATION OF A HARMONIC ELASTIC MEDIUM CONTAINING AN ELLIPSOIDAL CAVITY

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Abstract—This work is concerned with the formulation and solution of a problem for the response of a harmonic medium which, in its natural state, occupies the region exterior to an ellipsoid. The medium is deformed under the influence of pressure applied to the cavity wall, as well as stress applied at infinity.

1. INTRODUCTION

A class of equilibrium solutions is presented for finite deformation of an elastic medium containing an initially ellipsoidal cavity. The medium deforms under the combined influence of remotely applied stress and uniform pressure applied to the cavity wall. Because an inverse method is employed, these quantities cannot be assigned arbitrarily.

The medium under investigation in this study is of the *harmonic* type. Such materials are based upon a constitutive law of elastic response due to John[1] for which the partial differential equations governing finite elasticity are known to have certain advantages. In particular, harmonic materials have proven to be an attractive choice when exact analytical solutions are desired (see [1]–[6]). Harmonic materials are isotropic and homogeneous, and in contrast to the most frequently employed constitutive models, they allow for compressibility.

There have appeared a number of recent works dealing with the deformation of harmonic materials containing cavities. In [2], the plane-strain problem of a block containing a circular hole is treated for the case when the block is deformed by remotely applied uniaxial tension. Plane-strain is also dealt with in [3], but both the loading and the cavity configuration are more general than those of [2]. Abeyaratne and Horgan [6] have analyzed three-dimensional problems associated with the behavior of a hollow sphere, including the case where the outer radius tends to infinity, a case which furnishes a point of contact with the present article.

For incompressible materials, problems dealing with cylindrical cavities were first solved by Rivlin[7], in a work concerned with both inflation and stretching, whereas the problem of spherical inflation was solved by Green and Shield[8]. Both of these investigations concern deformations belonging to the class of deformations, sometimes called controllable or universal, which can be sustained in all homogeneous, isotropic, incompressible elastic materials without the application of body forces. In a recent study[9], Carroll has investigated load maximum instabilities associated with the inflation of incompressible hollow spheres and cylinders, both for general response and for special forms of the strain energy function.

In the next section, we summarize the equations governing finite deformations of a harmonic medium, and in Section 3 we present the solution of the problem. The solution is found in the form of a potential deformation, a restriction which somewhat limits its generality. It nevertheless enables us to make progress with a problem in which the loading and geometry are not restricted *a priori* to the usual symmetries.

2. GOVERNING EQUATIONS

Let R_0 denote the reference configuration and let R denote the final configuration. Denote by χ the function χ: $R_0 \rightarrow R$ that takes a point $\mathbf{X}$ from R_0 into its final position $\mathbf{x}$ in R,

$$\mathbf{x} = \chi(\mathbf{X}), \tag{2.1}$$

and let $\mathbf{F}$ stand for its gradient,

$$\mathbf{F} = \text{Grad}\ \chi. \tag{2.2}$$

Introduce the right polar decomposition of $\mathbf{F}$,

$$\mathbf{F} = \mathbf{RU}, \tag{2.3}$$

where $\mathbf{R}$, the rotation tensor, is orthogonal and $\mathbf{U}$, the right stretch tensor, is symmetric and positive definite. Let λ_k denote the principal stretches, so that λ_k are the roots of

$$\lambda^3 - i_1(\mathbf{U})\lambda^2 + i_2(\mathbf{U})\lambda - i_3(\mathbf{U}) = 0, \tag{2.4}$$

where $i_k(\mathbf{U})$ refer to the principal invariants of $\mathbf{U}$.

The notation $\mathbf{T}$ is used for the Cauchy stress, and $\mathbf{S}$ is employed for the Piola-Kirchhoff stress

$$\mathbf{S} = J\mathbf{T}\mathbf{F}^{-T},\ J = \det \mathbf{F}, \tag{2.5}$$

where $\mathbf{F}^{-T} = (\mathbf{F}^{-1})^T$. In the absence of body force, the equations of equilibrium appear as

$$\text{div}\ \mathbf{T} = 0,\ \mathbf{T} = \mathbf{T}^T,$$

or, in terms of the Piola-Kirchhoff stress,

$$\text{Div}\ \mathbf{S} = 0,\ \mathbf{SF}^T = \mathbf{FS}^T. \tag{2.6}$$

Surface force is expressed by the Cauchy relation

$$\mathbf{t} = \mathbf{Tn}, \tag{2.7}$$

where $\mathbf{n}$ stands for the unit normal to the surface at the point under consideration. The normal vector $\mathbf{n}$ is related to the normal $\mathbf{N}$ to the surface in its reference configuration through

$$\mathbf{n} = \mathbf{F}^{-T}\mathbf{N}/(\mathbf{N}\cdot\mathbf{F}^{-1}\mathbf{F}^{-T}\mathbf{N})^{1/2}. \tag{2.8}$$

For hyperelastic materials, there exists a stored energy function σ such that

$$\mathbf{S} = \sigma_{\mathbf{F}},\ \text{i.e.}\ S_{ij} = \partial\sigma/\partial F_{ij}. \tag{2.9}$$

In the case of an isotropic material, σ can be put in the form

$$\sigma = \hat{\sigma}(i_1(\mathbf{U}), i_2(\mathbf{U}), i_3(\mathbf{U})), \tag{2.10}$$

and as a consequence, eqn (2.9) furnishes

$$\mathbf{S} = (\hat{\sigma}_1 + i_1\hat{\sigma}_2)\mathbf{R} - \hat{\sigma}_2\mathbf{F} + i_3\hat{\sigma}_3\mathbf{F}^{-T}, \tag{2.11}$$

where

$$\hat{\sigma}_k = \partial\hat{\sigma}/\partial i_k. \tag{2.12}$$

For a *harmonic* material, $\hat{\sigma}$ has the special form[1]

$$\hat{\sigma} = G(i_1) + \alpha i_2 + \beta i_3, \tag{2.13}$$

where α and β are constants. Substituting into (2.11), we arrive at

$$\mathbf{S} = [G'(i_1) + \alpha i_1]\mathbf{R} - \alpha\mathbf{F} + \beta i_3 \mathbf{F}^{-T}. \tag{2.14}$$

We note that since Div $J\mathbf{F}^{-T} = 0$, the force balance $(2.6)_1$ reduces to

$$\mathrm{Div}\{[G'(i_1) + \alpha i_1]\mathbf{R} - \alpha\mathbf{F}\} = 0. \tag{2.15}$$

3. DEFORMATION OF A HARMONIC MEDIUM CONTAINING AN ELLIPSOIDAL CAVITY

Let X_i ($i = 1, 2, 3$) denote the components of the position vector of a point in the reference configuration relative to a fixed Cartesian frame, and let Ω denote the ellipsoid

$$\Omega = \left\{\mathbf{X}\left|\left(\frac{X_1}{A_1}\right)^2 + \left(\frac{X_2}{A_2}\right)^2 + \left(\frac{X_3}{A_3}\right)^2 \leqq 1\right.\right\}, \tag{3.1}$$

and let R_0 denote the complement of Ω.

Consider a harmonic medium having R_0 for its reference configuration. Our purpose is to describe a class of deformations which in the absence of body force are maintained in R by the application of a uniform pressure to ∂R and suitable stresses at infinity. This class of deformations lies within the class of *potential deformations*, i.e. those for which χ has the form*

$$\chi = \mathrm{grad}\ \Phi. \tag{3.2}$$

Here, we have

$$\mathbf{F} = \mathbf{U} = \mathrm{GradGrad}\ \Phi, \mathbf{R} = \mathbf{1}. \tag{3.3}$$

Accordingly, the equilibrium equation (2.15) becomes

$$\mathrm{Grad}[G'(i_1) + \alpha i_1 - \alpha\Delta\Phi] = 0, \tag{3.4}$$

where

$$\Delta\Phi = \mathrm{DivGrad}\ \Phi. \tag{3.5}$$

Further, it follows from $(3.3)_1$ that

$$i_1 = tr\mathbf{U} = \Delta\Phi, \tag{3.6}$$

so that eqn (3.4) further reduces to

$$G'(\Delta\Phi) = c. \tag{3.7}$$

where c is a constant. This is clearly satisfied if $\Delta\Phi$ is constant on R_0.

* John[10] uses the term "pseudo-irrotational."

Let ϕ denote the Newtonian potential of a homogeneous ellipsoid of unit density lying within Ω, i.e. let

$$\phi = \frac{1}{4\pi}\int_{\Omega}\frac{1}{|\mathbf{X}-\mathbf{Y}|}\,\mathrm{d}V_{\mathbf{Y}}. \tag{3.8}$$

It is well known that[11]

$$\Delta\phi = \begin{cases} -4\pi \text{ in the interior of } \Omega \\ 0 \text{ in } R_0 \end{cases} \tag{3.9}$$

and that ϕ can be put in the form[11, pp. 192–195]

$$\phi = \pi A_1 A_2 A_3 \int_{\xi}^{\infty}\left(\frac{X_1^2}{A_1^2+s}+\frac{X_2^2}{A_2^2+s}+\frac{X_3^2}{A_3^2+s}-1\right)\frac{\mathrm{d}s}{D(s)} \tag{3.10}$$

where

$$D(s) = \sqrt{(A_1^2+s)(A_2^2+s)(A_3^2+s)} \tag{3.11}$$

and ξ is determined by

$$\frac{X_1^2}{A_1^2+\xi}+\frac{X_2^2}{A_2^2+\xi}+\frac{X_3^2}{A_3^2+\xi} = 1 \tag{3.12}$$

Let $\mathbf{X}$ be a point of ∂R_0 and define

$$\text{GradGrad}\,\phi\,|_{\partial R_0}(\mathbf{X}) = \lim_{\substack{Y\to X\\ Y\in R_0}} \text{GradGrad}\,\phi(\mathbf{Y}) \tag{3.13}$$

Then, (3.10)–(3.12) furnish

$$\text{GradGrad}\,\phi\,|_{\partial R_0} = -4\pi(I_1\mathbf{e}_1\otimes\mathbf{e}_1 + I_2\mathbf{e}_2\otimes\mathbf{e}_2 + I_3\mathbf{e}_3\otimes\mathbf{e}_3 - \mathbf{N}\otimes\mathbf{N}), \tag{3.14}$$

where $\mathbf{e}_i$ denote the unit base vectors, I_k are defined by

$$I_k = \frac{A_1A_2A_3}{2}\int_0^{\infty}\frac{\mathrm{d}s}{(A_k^2+s)D}, \tag{3.15}$$

and $\mathbf{N}$ denotes the unit normal vector outward from R_0. Another important property is

$$\text{GradGrad}\,\phi(\mathbf{X}) \to 0 \text{ as } |\mathbf{X}| \to \infty, \tag{3.16}$$

which follows from (3.8).

Let a, b be constants, and set

$$\Phi = \frac{a}{2}|\mathbf{X}|^2 + \frac{b-a}{2}\left[\frac{1}{2\pi}\phi(\mathbf{X}) + I_1X_1^2 + I_2X_2^2 + I_3X_3^2\right]. \tag{3.17}$$

By (3.2), (3.3),

$$\chi = a\mathbf{X} + (b-a)\left[\frac{1}{4\pi}\text{Grad}\,\phi(\mathbf{X}) + I_1X_1\mathbf{e}_1 + I_2X_2\mathbf{e}_2 + I_3X_3\mathbf{e}_3\right], \tag{3.18}$$

$$\mathbf{U} = \mathbf{F} = a\mathbf{1} + (b-a)\left[\frac{1}{4\pi}\text{GradGrad}\,\phi(\mathbf{X}) + I_1\mathbf{e}_1\otimes\mathbf{e}_1 + I_2\mathbf{e}_2\otimes\mathbf{e}_2 + I_3\mathbf{e}_3\otimes\mathbf{e}_3\right]. \tag{3.19}$$

Therefore, and by eqn (3.9),

$$\Delta\Phi = 2a + b \tag{3.20}$$

since[11, pp. 194–195]

$$l_1 + l_2 + l_3 = 1. \tag{3.21}$$

Thus, with Φ given by eqn (3.17), the equilibrium condition (3.7) is fulfilled.

Our next task is to determine the stress induced by the deformation (3.18). By eqns $(2.5)_1$, (2.14), $(3.3)_2$, (3.6), and (3.20), the Cauchy stress $\mathbf{T}$ is given by

$$\mathbf{T} = \frac{1}{J}\{[G'(2a + b) + \alpha(2a + b)]\mathbf{F} - \alpha\mathbf{F}^2\} + \beta\mathbf{1}. \tag{3.22}$$

From eqns (3.19), (3.16), there follows

$$\lim_{|\mathbf{X}|\to\infty} \mathbf{F}(\mathbf{X}) = [a + (b - a)l_1]\mathbf{e}_1 \otimes \mathbf{e}_1 + [a + (b - a)l_2]\mathbf{e}_2 \otimes \mathbf{e}_2 + [a + (b - a)l_3]\mathbf{e}_3 \otimes \mathbf{e}_3. \tag{3.23}$$

Accordingly, for $i \neq j$, (3.22) furnishes

$$\lim_{|\mathbf{X}|\to\infty} T_{ij} = 0, \tag{3.24}$$

and if we set

$$\overset{\infty}{T}_k = \lim_{|\mathbf{X}|\to\infty} T_{kk} \quad \text{(no sum)}, \tag{3.25}$$

there follows

$$\overset{\infty}{T}_k = \frac{G'(2a + b) + \alpha(2a + b) - \alpha[a + (b - a)l_k]}{\prod_{l\neq k}[a + (b - a)l_l]} + \beta. \tag{3.26}$$

We turn now to the surface tractions required to maintain the deformation (3.18). In view of eqn (3.14) and (3.19),

$$\mathbf{F}\,|_{\partial R_0} = a(\mathbf{1} - \mathbf{N}\otimes\mathbf{N}) + b\mathbf{N}\otimes\mathbf{N}, \tag{3.27}$$

from which it is evident that the stretch normal to the surface has the constant value b, while the tangential stretches are independent of direction and have the constant value a. Thus,

$$J\,|_{\partial R_0} = a^2 b, \tag{3.28}$$

and

$$\mathbf{F}^2\,|_{\partial R_0} = a^2(\mathbf{1} - \mathbf{N}\otimes\mathbf{N}) + b^2\mathbf{N}\otimes\mathbf{N}. \tag{3.29}$$

By eqns (3.22), (3.27), and (3.28), at points of ∂R,

$$\mathbf{T} = \frac{1}{a^2 b}\{[G'(2a + b) + \alpha(2a + b)][a(\mathbf{1} - \mathbf{N}\otimes\mathbf{N}) + b\mathbf{N}\otimes\mathbf{N}] - \alpha[a^2(\mathbf{1} - \mathbf{N}\otimes\mathbf{N}) + b^2\mathbf{N}\otimes\mathbf{N}]\} + \beta\mathbf{1}. \tag{3.30}$$

From (3.27), we see that

$$\mathbf{F}^{-T}\,|_{\partial R_0} = \mathbf{F}^{-1}\,|_{\partial R_0} = \frac{1}{a}(\mathbf{1} - \mathbf{N}\otimes\mathbf{N}) + \frac{1}{b}\mathbf{N}\otimes\mathbf{N}. \tag{3.31}$$

Accordingly, and in view of eqn (2.8), there follows

$$\mathbf{n} = \mathbf{N}. \tag{3.32}$$

Hence, eqn (3.30) furnishes

$$\mathbf{t} = -p\mathbf{n}, \tag{3.33}$$

where p is the *constant*

$$p = -\frac{1}{a^2}[G'(2a + b) + 2\alpha a] - \beta. \tag{3.34}$$

Our goal for the present work has now been realized, but of course much remains to be done. The results in [6, 9] for cylindrical and spherical geometries suggest several questions for the stability of the states found here. In particular, what properties should be ascribed to the response function G in order to ensure physically reasonable behavior? Another aspect that ought to be investigated is the connection between the present results and those obtained in [3] for cylindrical cavities of elliptical cross-section. In what sense do the present results tend to those of [3] as, say, $A_3 \to \infty$?

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